Large-Scale and Large-Time Behaviour of Mean-Field Interacting Particle Systems on Block-structured Networks

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at 16th Workshop on Markov Processes and Related Topics
July 16, 2021

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Overview

1. Mean-field model: The homogeneous case
2. Mean-field Models: Heterogeneous case
3. Large scale behavior
4. Large time behavior

Outline

Interacting particle systems and mean-field approach

- The interest in IPS started with statistical physics
- Pioneer works: Boltzmann, Vlasov, Curie, Wisse, Ising, and others
- The purpose: Understand the global (average) behavior of a very large number of particles interacting with each other
- The mean-field approach aims to obtain a smaller object through an average over the interactions (dimension reduction!)
- Pioneer work: McKean [1966] studied the mean-field approach for interacting diffusions

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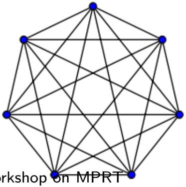
Classical mean-field model

- N symmetric (in distribution) particles interacting with each other
- The state space of each particle is $\mathcal{Z}$: Discrete or Continuous
- $X_n^N(t)$: state of the nth particle at time t (a Markov chain)
- Due to symmetry of the particles, to describe the system, it is enough to use the coupled dynamics, or the empirical distribution of all particles across states:

$$\mu^N(t) = \frac{1}{N}\sum_{n=1}^{N} \delta_{X_n^N(t)} \in \mathcal{M}_1(\mathcal{Z}), \quad \text{space of prob. measures on } \mathcal{Z}$$

$\Rightarrow$ Gives the fraction of particles in each subset of $\mathcal{Z}$.

- A picture of global interactions:



Classical Mean-field model: classical results

- Example of the state space: if $\mathcal{Z} = \{1, \ldots, K\}$ (finite state)
- Let $\Lambda(\mu^N(t)) = (\lambda_{z,z'}(\mu^N(t)))_{(z,z')\in\mathcal{Z}\times\mathcal{Z}}$ be the rate matrix over $\mathcal{Z}$ (depends on the empirical measure!)
- Law of Large numbers: for each $T > 0$, $\mu^N(\cdot) \to \mu(\cdot)$ in probability uniformly on $[0, T]$, where $\mu(\cdot)$ solves the McKean-Vlasov equation

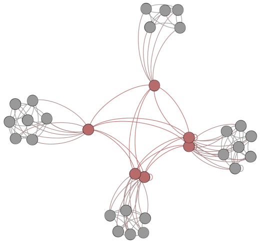
$$\dot{\mu}(t) = \Lambda(\mu(t))^* * \mu(t),$$
$$\mu(0) = \nu$$

  where the coefficients of the SDE depend on the distribution of the solution
- Propagation of chaos: if the particles are initially iid, and we tag finite k particles, then their evolution is asymptotically iid over any finite time interval!
- **Consequence:** the study of one particle gives information on the behavior of the population

Outline

What if the interaction graph is not complete?

- Suppose the interaction graph is not complete, i.e. not all particles interact with each other!
- Things get more complicated! Why: we lose the global symmetry between particles
- What to do: detect local symmetries and average around them!
- Special case of interest: block-structured graphs!



The model: Structure of the graph

- Consider a block-structured graph $\mathcal{G} = (\mathcal{V}, \Xi)$, composed of r blocks (populations) and each node representing a particle
- Each block C_j is a clique, i.e. all the N_j nodes are connected to each other
- The nodes of each block C_j are divided into two categories:
  - **Central nodes** C_j^c**:** connected only to the nodes of the same block
  - **Peripheral nodes** C_j^p**:** connected to the nodes of the same block and also to peripheral nodes of the other blocks
- We have $card(C_j^c) = N_j^c$ and $card(C_j^p) = N_j^p$

The model dynamic

- Finite state space: $\mathcal{Z} = \{1, 2, \ldots, K\} \subset \mathbb{N}$ (colors)
- For each block j, $(X^c_{n,j}(t), t \geq 0)$ is the color of central (particle) node n at time t; $(X^p_{n,j}(t), t \geq 0)$ is the color of peripheral node n at time t
- We characterize the system's state by local empirical measures:

$$\mu^{c,N}_j(t) = \frac{1}{N^c_j} \sum_{n \in C^c_j} \delta_{X^c_{n,j}(t)}$$

$$\mu^{p,N}_j(t) = \frac{1}{N^p_j} \sum_{n \in C^p_j} \delta_{X^p_{n,j}(t)}$$

- Fix a block $1 \leq j \leq r$:
  - The neighborhood's state of $n \in C^c_j$ is characterized by $\mu^{c,N}_j(t)$, $\mu^{p,N}_j(t)$ (only nodes in the same block)
  - The neighborhood's state of $n \in C^p_j$ is characterized by $\mu^{c,N}_j(t)$; $\mu^{p,N}_1(t)$, $\mu^{p,N}_2(t)$, $\ldots$, $\mu^{p,N}_r(t)$ (nodes in the same block and peripheral nodes in other blocks)

The model dynamic

- **The central nodes transitions.** For $n \in C_j^c$, its color $X_{n,j}^c(t)$ goes from z to z' at rate:

$$\lambda_{z,z'}^c\big(\mu_j^{c,N}(t), \mu_j^{p,N}(t)\big)$$

- **The peripheral nodes transitions.** For node $n \in C_j^p$, its color $X_{n,j}^p(t)$ transits from z to z' at rate:

$$\lambda_{z,z'}^p\big(\mu_j^{c,N}(t), \mu_1^{p,N}(t), \mu_2^{p,N}(t), \dots, \mu_r^{p,N}(t)\big)$$

- Some additional notations:
  - $\mathcal{D}([0,T],\mathcal{Z})$ the Skorokhod space of *cadlag* functions from $[0,T]$ to $\mathcal{Z}$
  - $\mathcal{M}_1(\mathcal{D}([0,T],\mathcal{Z}))$ the set of probability measures on $\mathcal{D}([0,T],\mathcal{Z})$

SDE representation of the system

- The Markov chains $X^c_{n,j}$ and $X^p_{n,j}$ can be represented by the following system of SDE's

$$X^c_{n,j}(t) = X^c_{n,j}(0) + \int_{[0,t]\times\mathbb{R}_+} \sum_{(z,z')\in\mathcal{E}} \mathbb{1}_{X^c_{n,j}(s-)=z}(z'-z)\mathbb{1}_{\left[0,\lambda^c_{z,z'}\left(\mu^{c,N}_j(s-),\mu^{p,N}_j(s-)\right)\right]}(y)\mathcal{N}^c_{n,j}(ds,dy)$$

$$X^p_{n,j}(t) = X^p_{n,j}(0) + \int_{[0,t]\times\mathbb{R}_+} \sum_{(z,z')\in\mathcal{E}} \mathbb{1}_{X^p_{n,j}(s-)=z}(z'-z)\mathbb{1}_{\left[0,\lambda^p_{z,z'}\left(\mu^{c,N}_j(s-),\mu^{p,N}_1(s-),\dots,\mu^{p,N}_r(s-)\right)\right]}(y)\mathcal{N}^p_{n,j}(ds,dy)$$

where $\{\mathcal{N}^c_{n,j}, n\in C^c_j, 1\le j\le r\}$ and $\{\mathcal{N}^p_{n,j}, n\in C^p_j, 1\le j\le r\}$ are collections of Poisson random measures on $\mathbb{R}^2$ whose intensity measure is the Lebesgue measure on $\mathbb{R}^2_+$

Outline

Large-scale behavior: Multi-chaoticity

Recall: Propagation of chaos means that the stochastic independence of fixed k particles persists as the number of particles goes to infinity

Theorem

Suppose that the initial conditions converge in distribution towards $\nu^{1,c} \otimes \nu^{1,p} \cdots \nu^{r,c} \otimes \nu^{r,p}$. Therefore, under some regularity conditions, the propagation of chaos (in multi-populations) holds over any finite interval of time, i.e. for any $k \geq 1$,

$$\lim_{N\to\infty}(X^c_{n,j}, X^p_{n,j}, 1 \leq n \leq k, 1 \leq j \leq r)) \stackrel{dist}{=} (\mu^c_1)^k \otimes (\mu^p_1)^k \cdots (\mu^c_r)^k \otimes (\mu^p_r)^k$$

holds for the topology of the uniform convergence on compact sets, where $\mu = \mu^c_1 \otimes \mu^p_1 \cdots \mu^c_r \otimes \mu^p_r$ is the distribution of the process $\big((\bar{X}^c_{n,j}(t), \bar{X}^p_{m,j}(t), t \geq 0), n \in C^c_j, m \in C^p_j; 1 \leq j \leq r\big)$, solution of a limiting SDE with initial distribution $\nu^{1,c} \otimes \nu^{1,p} \cdots \nu^{r,c} \otimes \nu^{r,p}$

The limiting process $\left((\bar{X}^c_{n,j}(t), \bar{X}^p_{n,j}(t), t \in [0, T]), 1 \leq j \leq r\right)$ is solution of the following system of SDE's

$$\bar{X}^c_{n,j}(t) = \bar{X}^c_{n,j}(0) + \int_{[0,t]\times\mathbb{R}_+} \sum_{(z,z')\in\mathcal{E}} \mathbb{1}_{\bar{X}^c_{n,j}(s-)=z}(z'-z)\mathbb{1}_{\left[0,\lambda^c_{z,z'}\left(\mu^c_j(s-),\mu^p_j(s-)\right)\right]}(y)\mathcal{N}^c_{n,j}(ds,dy),$$

$$\bar{X}^p_{n,j}(t) = \bar{X}^p_{n,j}(0) + \int_{[0,t]\times\mathbb{R}_+} \sum_{(z,z')\in\mathcal{E}} \mathbb{1}_{\bar{X}^p_{n,j}(s-)=z}(z'-z)\mathbb{1}_{\left[0,\lambda^p_{z,z'}\left(\mu^c_j(s-),\mu^p_1(s-),\ldots,\mu^p_r(s-)\right)\right]}(y)\mathcal{N}^p_{n,j}(ds,dy)$$

where

$$\mu = \left(\mu^c_1, \mu^p_1, \cdots, \mu^c_r, \mu^p_r\right) = \left(\mathcal{L}(\bar{X}^c_{n,1}), \mathcal{L}(\bar{X}^p_{n,1}), \ldots, \mathcal{L}(\bar{X}^c_{n,r}), \mathcal{L}(\bar{X}^p_{n,r})\right) \in \left(\mathcal{M}_1(\mathcal{D}([0,T],\mathcal{Z}))\right)^{2r},$$

Large-scale behavior: Laws of Large Numbers

- As a consequence of the propagation of chaos result, we obtain laws of large numbers for the local empirical measures

Corollary (LLN)

Denote $\mu_j^c = \mathcal{L}(\bar{X}_{n,j}^c), \mu_j^p = \mathcal{L}(\bar{X}_{n,j}^p)$ *for* $1 \leq j \leq r$*, then, as* $N \to \infty$,

$$\mu_j^{c,N} = \frac{1}{N_j^c} \sum_{n \in C_j^c} \delta_{X_{n,j}^c} \to \mu_j^c \quad in \quad \mathcal{M}_1(\mathcal{D}([0,T],\mathcal{Z})) \quad in\ probability,$$

$$\mu_j^{p,N} = \frac{1}{N_j^p} \sum_{n \in C_j^p} \delta_{X_{n,j}^p} \to \mu_j^p \quad in \quad \mathcal{M}_1(\mathcal{D}([0,T],\mathcal{Z})) \quad in\ probability,$$

Law of Large numbers: Consequence

From the LLN, we deduce that, as $N \to \infty$, the sequence $(\mu^N = (\mu_1^{c,N}, \mu_1^{p,N}, \ldots, \mu_r^{c,N}, \mu_r^{p,N}))$ converges weakly towards the solution μ of the McKean-Vlasov system

$$\begin{cases} \dot{\mu}_j^c(t) = A^*_{(\mu_j^c(t), \mu_j^p(t))} \mu_j^c(t), \\ \dot{\mu}_j^p(t) = A^*_{(\mu_j^c(t), \mu_1^p(t), \ldots, \mu_r^p(t))} \mu_j^p(t), \\ \mu_j^c(0) = \nu_j^c, \mu_j^p(0) = \nu_j^p, \\ 1 \le j \le r, t \in [0, T], \end{cases} \tag{1}$$

where A^* is the adjunct/transpose of A, and

$$A_{\mu_j^c(t), \mu_j^p(t)} = \Big(\lambda_{z,z'}^c(\mu_j^c(t), \mu_j^p(t))\Big)_{(z,z') \in \mathcal{Z} \times \mathcal{Z}}$$

is the rate matrix for central nodes in block j, and

$$A_{\mu_j^c(t), \mu_1^p(t), \ldots, \mu_r^p(t)} = \Big(\lambda_{z,z'}^p(\mu_j^c(t), \mu_1^p(t), \ldots, \mu_r^p(t))\Big)_{(z,z') \in \mathcal{Z} \times \mathcal{Z}},$$

is the rate matrix for peripheral nodes in block j

Outline

Large time behavior: A high level picture

- From LLN, as $N \to \infty$,

$$\mu^N(t) = (\mu_j^{N,c}(t), \mu_j^{N,p}(t), 1 \le j \le r) \to \mu(t) = (\mu_j^c(t), \mu_j^p(t), 1 \le j \le r)$$

- Thus:

$$\lim_{t\to\infty} \left[\lim_{N\to\infty} \mu^N(t) \right] \to \lim_{t\to\infty} \left[\mu(t) \right]$$

$\Rightarrow$ amount to a study the McKean-Vlasov system

- What about $\lim_{N\to\infty} \left[\lim_{t\to\infty} \mu^N(t) \right]$?
  - For N fixed: if μ^N is irreducible then there exists a unique stationary distribution $\wp^N$ for μ^N
  - What happened for $\wp^N$ when $N \to \infty$?
    $\Rightarrow$ Study the large deviations of $(\wp^N, N \ge 1)$

LDP for the stationary distribution

- Two distinct scenarios depending on the large time behavior of the McKean-Vlasov system:
  - **Unique globally asymptotically stable equilibrium** ξ_0: one might prove that $\wp^N \to \delta_{\xi_0}$, i.e. $\mu^N(\infty) \to \xi_0$ in distribution
  - **Multiple ω-limit sets**: which of these characterize the limiting behavior of μ^N?
    In this case we assume that there exist a finite number of compact sets K_1, K_2 ,..., K_ℓ such that every ω-limit set of the McKean-Vlasov system lies completely in one of the compact sets K_i. (Hypothesis of Freidlin-Wantzell).

Case 1: Unique GAS equilibrium ξ_0

Theorem

If the McKean-Vlasov equation has a unique globally asymptotically stable equilibrium ξ_0, then the sequence $(\wp^N, N \geq 1)$ satisfies a LDP with speed N and a good rate function s given by

$$s(\xi) = \inf_{\hat{\mu}} \sum_{j=1}^{r} \Bigg[\alpha_j p_j^c \int_0^{+\infty} \Bigg(\sum_{(z,z')\in\mathcal{E}} (\hat{\mu}_j^c(t)(z)) \lambda_{z,z'}^c(\cdot)\, \tau^* \Bigg(\frac{\hat{l}_{z,z'}^{j,c}(t)}{\lambda_{z,z'}^c(\cdot)} - 1 \Bigg) \Bigg) dt$$

$$+ \alpha_j p_j^p \int_0^{+\infty} \Bigg(\sum_{(z,z')\in\mathcal{E}} (\hat{\mu}_j^p(t)(z)) \lambda_{z,z'}^p(\cdot)\, \tau^* \Bigg(\frac{\hat{l}_{z,z'}^{j,p}(t)}{\lambda_{z,z'}^p(\cdot)} - 1 \Bigg) \Bigg) dt \Bigg]$$

where the infimum is over all the infinite paths $\hat{\mu}$ that are solutions to the reversed-time dynamical system

$$\dot{\hat{\mu}}_j^c(t) = -\hat{L}_{j,c}(t)^* \hat{\mu}_j^c(t),$$
$$\dot{\hat{\mu}}_j^p(t) = -\hat{L}_{j,p}(t)^* \hat{\mu}_j^p(t),$$

for some family of rate matrices $\hat{L}_{j,c}$ and $\hat{L}_{j,p}$, with initial condition $\mu(0) = \xi$, terminal condition $\lim_{t\to\infty} \mu(t) = \xi_0$, and $\mu(t) \in (\mathcal{M}_1(\mathcal{Z}))^{2r}$ for all $t \geq 0$.

The intuition behind the previous result

- From LDP of $\wp^N$, for a given $\xi \in (\mathcal{M}_1(\mathcal{Z}))^{2r}$,

$$\mathbb{P}(\mu^N(+\infty) \approx \xi) \approx \exp(-Ns(\xi)), \quad \text{as } N \to +\infty$$

  $\Rightarrow$ The rate function s characterizes the "difficulty" of the passage of $\mu^N(+\infty)$ near ξ

- Interpretation of previous theorem: if $\mu^N(+\infty)$ is near ξ, then this is most likely due to a trajectory that began at ξ_0, worked against the attractor ξ_0, and took the lowest cost path $\hat{\mu}$ to ξ over all possible time duration

Case 2: Multiple ω-limit sets

- Under Freidlin-Wantzell hypothesis: We obtain a similar result but now we also take the infimum over all the compact sets K_i!

Case 2: Multiple ω-limit sets (Cont'd)

Theorem

The sequence of stationary distributions $(\wp^N, N \geq 1)$ satisfies the LDP with speed N and a good rate function s given by

$$s(\xi) = \inf_{l'} \inf_{\hat{\mu}} \Bigg[s_{l'} + \sum_{j=1}^{r} \Bigg[\alpha_j p_j^c \int_0^{+\infty} \Bigg(\sum_{(z,z')\in\mathcal{E}} (\hat{\mu}_j^c(t)(z)) \lambda_{z,z'}^c(\cdot)\, \tau^* \Bigg(\frac{\hat{l}_{z,z'}^{j,c}(t)}{\lambda_{z,z'}^c(\cdot)} - 1 \Bigg) \Bigg) dt$$
$$+ \alpha_j p_j^p \int_0^{+\infty} \Bigg(\sum_{(z,z')\in\mathcal{E}} (\hat{\mu}_j^p(t)(z)) \lambda_{z,z'}^p(\cdot)\, \tau^* \Bigg(\frac{\hat{l}_{z,z'}^{j,p}(t)}{\lambda_{z,z'}^p(\cdot)} - 1 \Bigg) \Bigg) dt \Bigg] \Bigg]$$

where the constants $s_{l'}$ determine the "difficulty" of passage from one compact set to another, and the second infimum is over all $\hat{\mu}$ that are solutions to the reversed-time dynamical system

$$\dot{\hat{\mu}}_j^c(t) = -\hat{L}_{j,c}(t)^* \hat{\mu}_j^c(t),$$
$$\dot{\hat{\mu}}_j^p(t) = -\hat{L}_{j,p}(t)^* \hat{\mu}_j^p(t),$$

for some family of rate matrices $\hat{L}_{j,c}$ and $\hat{L}_{j,p}$, with initial condition $\mu(0) = \xi$, terminal condition $\lim_{t\to\infty} \mu(t) \in K_{l'}$, and $\mu(t) \in (\mathcal{M}_1(\mathcal{Z}))^{2r}$ for all $t \geq 0$.

Phenomena from one ω-limit set to another

- Let's summarize:
  - From LLN: as $N \to \infty$, $\mu^N \to \mu \Rightarrow$ Use McKean-Vlasov equation to study the large t behavior
  - As $t \to \infty$, $\mathcal{L}(\mu^N(\infty)) = \wp^N \Rightarrow$ Use LDP results to study large N behavior of $\wp^N$
- What about: $\lim_{t\to\infty} \mu^N(t)$ for large but finite N?
  $\Rightarrow$ If multiple ω-limit sets for McKean-Vlasov, we observe metastable phenomena
- Metastable behavior: transitions and exit times from one ω-limit set to another!
  $\Rightarrow$ Freidlin-Wentzell approach: rely on the study of an embedded Markov chain of states at hitting times of neighborhood of the ω-limit sets

Metastable phenomena: some ideas

- Adapting the Freidlin-Wentzell approach: view the finite N system μ^N as a small noise perturbation of the deterministic μ solution of the McKean-Vlasov system
  $\Rightarrow N^{-1}$ plays the role of the "small noise" parameter ε of Freidlin-Wentzell
- Examples of obtained estimates:
  - The mean time spent by μ^N near an ω-limit set,
  - The probability of reaching a given ω-limit set before reaching another one,
  - The probability of traversing a collection of ω-limit sets in a particular order (limit cycles)...

Some interesting questions

- How to numerically compute the rate functions characterizing the LDP of the stationary distributions $\wp^N$
  $\Rightarrow$ Seems to be challenging even in the simpler complete graph context! [Borkar et al.]
- Study the stability properties of the McKean-Vlasov equation
  $\Rightarrow$ Possible approach: identifying the limit of relative entropies w.r.t $\wp^N$ as a possible Lyapunov function [Budhiraja et al.]

References

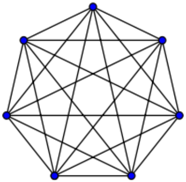
- D.A. Dawson, A. Sid-Ali, Y.Q. Zhao, *'Propagation of chaos and large deviations in mean-field models with jumps on block-structured networks'*. Submitted. arXiv:2012.02870 [math.PR]
- D.A. Dawson, A. Sid-Ali, Y.Q. Zhao, *'Large time behavior for a mean-field model with jumps on block-structured networks'*. In preparation
- V.S. Borkar and R. Sundaresan. *'Asymptotics of the invariant measure in mean field models with jumps.'* Stochastic Systems. 2012
- M.I. Freidlin and A.D Wentzell. *'Random Perturbations of Dynamical Systems.'* Springer Berlin Heidelberg, 2012.
- A. Budhiraja, P. Dupuis, M. Fischer, and K. Ramanan. *'Local stability of Kolmogorov forward equations for finite state nonlinear Markov processes.* Electron. J. Probab. 2015.
- S. Yasodharan and R. Sundaresan. *'Large time behaviour and the second eigenvalue problem for finite state mean-field interacting particle systems.'* arXiv:1909.03805 [math.PR], 2019.

Thank you for listening!

Any questions?

Appendix 1: Classical Mean-field model

- Take e.g. $\mathcal{Z} = \{1, \ldots, K\}$
- Transition rate matrices $\Lambda(\mu^N(t)) = (\lambda_{z,z'}(\mu^N(t)))_{(z'z)\in\mathcal{Z}^2}$, for some (Lipschitz) functions $\lambda_{z,z'}$ on $\mathcal{M}_1(\mathcal{Z})$
- Consider the Markov process $(X_n(\cdot), 1 \leq n \leq N)$: its state space is K^N $\Rightarrow$ Exponential growth!
- Alternative idea: track the measure-valued Markov process $\mu_N(\cdot)$ instead: its state space size is of order at most $(N+1)^K \Rightarrow$ Draw conclusions on the original process



Appendix 1: Classical Mean-field model- Law of large numbers

Theorem (Kurtz)

Under some regularity assumptions, if $\mu_N(0) \to \nu$ in probability, then for each $T > 0$, $\mu_N(\cdot) \to \mu(\cdot)$ in probability uniformly on $[0, T]$, where $\mu(\cdot)$ solves the McKean-Vlasov equation

$$\dot{\mu}(t) = \Lambda(\mu(t))^* * \mu(t),$$
$$\mu(0) = \nu$$

N.B. $\mu_N(\cdot) \in \mathcal{D}([0, T], \mathcal{M}_1(\mathcal{Z}))$ equipped with the metric

$$\rho_T(\mu, \nu) = \sup_{0 \le t \le T} \rho_0(\mu_t, \nu_t),$$

where $\rho_0(\alpha, \beta)$ generates the weak topology on $\mathcal{M}_1(\mathcal{Z})$, e.g. bounded-Lipschitz metric, L_1 metric...

Appendix 1- Classical Mean-field model-Propagation of chaos

- Let $N \to \infty$, thus $\mu_N(\cdot) \to \mu(\cdot)$ solution of McKean-Vlasov
- Tag a particle in the limit: its evolution is described asymptotically by a Markov process with rates $\Lambda_{z,z'}(\mu(t))$
  $\Rightarrow$ At t, it is in state z with probability $\mu(t)(z)$
- Tag k particles:
  - If $(X_n(0), 1 \leq n \leq N)$ are exchangeable and $\mu_N(0) \to \nu$ in probability, then their states are asymptotically independent at time 0
  - Thanks to the LLN, the evolution is iid across the particles
- Thus: the "chaos" (independence) propagates in time!
- Consequence: the study of one individual gives information on the behavior of the group the group

N.B. POC and LLN are here equivalent. See, e.g. [Sznitman]

Appendix 2: LDP from the McKean-Vlasov system over finite $[0, T]$

Theorem

Denote $p^N_{\nu_N} = \mathcal{L}(\mu^N)$. Suppose that $\nu_N \to \nu$ weakly. The sequence $(p^N_{\nu_N}, N \geq 1)$ obeys a LDP with speed N, and a good rate function $S_{[0,T]}(\mu|\nu)$. Moreover, if a path μ satisfies $S_{[0,T]}(\mu|\nu) < \infty$, then there exist rate families $(l^{j,c}_{z,z'}(t), t \in [0, T])$ and $(l^{j,p}_{z,z'}(t), t \in [0, T])$ such that, for all $1 \leq j \leq r$,

$$\dot{\mu}^c_j(t) = L_{j,c}(t)^* \mu^c_j(t),$$
$$\dot{\mu}^p_j(t) = L_{j,p}(t)^* \mu^p_j(t),$$

where $L_{j,c}(t)$, $L_{j,p}(t)$ are the rate matrices associated with the time-varying rates $(l^{j,c}_{z,z'}(t))$, $(l^{j,p}_{z,z'}(t)$ and $L_{j,c}(t)^*$. Furthermore, in this case

$$S_{[0,T]}(\mu|\nu) = \sum_{j=1}^{r} \Bigg[\alpha_j p^c_j \int_0^T \Bigg(\sum_{(z,z')\in\mathcal{E}} (\mu^c_j(t)(z)) \lambda^c_{z,z'}(\cdot)\, \tau^* \Bigg(\frac{l^{j,c}_{z,z'}(t)}{\lambda^c_{z,z'}(\cdot)} - 1 \Bigg) \Bigg) dt$$
$$+ \alpha_j p^p_j \int_0^T \Bigg(\sum_{(z,z')\in\mathcal{E}} (\mu^p_j(t)(z)) \lambda^p_{z,z'}(\cdot)\, \tau^* \Bigg(\frac{l^{j,p}_{z,z'}(t)}{\lambda^p_{z,z'}(\cdot)} - 1 \Bigg) \Bigg) dt \Bigg].$$

What the previous theorem tells us?

- From LDP of $p^N_{\nu_N}$, for a given path μ,

$$\mathbb{P}(\mu^N = \mu) \approx \exp(-NS_{[0,T]}(\mu|\nu)), \quad \text{as } N \to +\infty$$

$\Rightarrow$ The action functional S characterizes the "difficulty" of the passage of μ^N near μ in the time interval $[0, T]$

- If $S_{[0,T]}(\mu|\nu) = 0$, then μ must be the solution to the McKean-Vlasov equation with initial condition $\mu(0) = \nu$ (the Legendre transform satisfies $\tau^*(0) = 0$)
  $\Rightarrow$ The McKean-Vlasov path has zero "cost"
- From LDP of the empirical measure we can investigate the LDP of the stationary distribution...

Quasipotential

- Important notion: the quasipotential defined for any $\nu, \xi \in (\mathcal{M}_1(\mathcal{Z}))^{2r}$ as

$$V(\xi|\nu) = \inf\{S_{[0,T]}(\mu|\nu) : \mu(0) = \nu, \mu(T) = \xi, T > 0\},$$

$\Rightarrow$ Measures the "difficulty" for the empirical process to move from ν to ξ in finite time

Indices characterizing the passage through compacts sets

- Take $L = \{1, 2, \ldots, l\}$ the indices corresponding to the compact sets $K_1, K_2, \ldots, K_l$
- The rate $s_{l'}$, $1 \leq l' \leq l$ are given by $s_{l'} = W(K_{l'}) - \min_{l'} W(K_{l'})$, where

$$W(K_i) = \min_{g \in G\{i\}} \sum_{(i,j) \in g} V(K_i, K_j)$$

with $G\{i\}$ is the W-graph corresponding to $W = i$, with $i \in \{1, \ldots, l\}$.

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